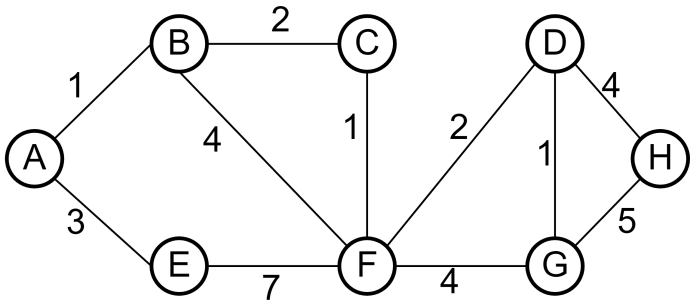
CS 61B Spring 2024

Shortest Paths, MSTs

Exam-Level 09: March 18, 2024

1 Dijkstra's, A*

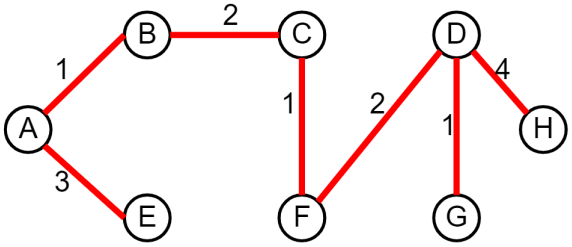


(a) Run Dijkstra's Algorithm on the graph above starting from vertex A, breaking ties alphabetically. Fill in how the priority values change below. When you remove a node from the fringe, mark it with a check, and leave it blank for the subsequent rows. Stop when you remove G. Also sketch the resulting shortest paths tree in the end.

Solution:

Node	A	B	C	D	E	F	G	H
Start	0	∞	∞	∞	∞	∞	∞	∞
Iter 1	✓	1	∞	∞	3	∞	∞	∞
Iter 2		✓	3	∞	3	5	∞	∞
Iter 3			✓	∞	3	4	∞	∞
Iter 4				∞	✓	4	∞	∞
Iter 5				6		✓	8	∞
Iter 6				✓			7	10
Iter 7							✓	

The SPT of the whole graph is drawn below for your reference. You should be able to "read-off" the `edgeTo` pointers by locating the last time the priority value of a node changes, and search for where the "checkmark" is for that row - that row would be the edge you came from in the SPT!



(b) The heuristic distance from all nodes to G is defined below. Run A*, starting from A and with G as a goal. For each entry in the table below, fill in the distance, followed by the priority value of each node,

separated by a comma. Is the heuristic admissible?

u	A	B	C	D	E	F	G	H
$h(u, G)$	9	7	4	1	10	3	0	5

Node	A	B	C	D	E	F	G	H
Start	0, 9	∞	∞	∞	∞	∞	∞	∞
Iter 1	✓							
Iter 2								
Iter 3								
Iter 4								
Iter 5								
Iter 6							✓	

Solution:

Node	A	B	C	D	E	F	G	H
Start	0, 9	∞	∞	∞	∞	∞	∞	∞
Iter 1	✓	1, 8	∞	∞	3, 13	∞	∞	∞
Iter 2		✓	3, 7	∞	3, 13	5, 8	∞	∞
Iter 3			✓	∞	3, 13	4, 7	∞	∞
Iter 4				6, 7	3, 13	✓	8, 8	∞
Iter 5				✓	3, 13		7, 7	10, 15
Iter 6							✓	

The shortest path is: $A \to B, B \to C, C \to F, F \to D, D \to G$.

This heuristic is not admissible ($h(A, G) > dist(A, G) = 7$). However, in this graph A* still returns the correct shortest path from A to G (recall in lecture, admissible and consistent heuristic is only a sufficient condition, so you can still have inadmissible heuristics and still correct results).

2 Conceptual Shortest Paths

Answer the following questions regarding shortest path algorithms for a **weighted, undirected graph**. If the statement is true, provide an explanation. If the statement is false, provide a counterexample.

(a) (T/F) If all edge weights are equal and positive, the breadth-first search starting from node A will return the shortest path from a node A to a target node B.

Solution: True. If all edges are equal in weight, then the shortest path from A to each node is proportional to the number of nodes on the path, so breadth first search will return the shortest path.

(b) (T/F) If all edges have distinct weights, the shortest path between any two vertices is unique.

Solution: False. Consider a case of 3 nodes where AB is 3, AC is 5, and BC is 2. Here, the two possible paths from A to C both are of length 5.

(c) (T/F) **Adding** a constant positive integer k to all edge weights will not affect any shortest path between two vertices.

Solution: False. Consider a case of 3 nodes A, B, and C where AB is 1, AC is 2.5 and BC is 1. Clearly, the best path from A to C is through B, with weight 2. However, if we add 1 to each edge weight, suddenly the path going through B will have weight 4, while the direct path is only 3.5.

(d) (T/F) **Multiplying** a constant positive integer k to all edge weights will not affect any shortest path between two vertices.

Solution: True. Suppose we have arbitrary nodes u and v. Let's say the shortest path from u to v, before the multiplication by k, was of total weight w. This implies that every other path from u to v was of total weight greater than w. After multiplying each edge weight by k, the total weight of the shortest path becomes $w * k$ and the total weight of every other path becomes some number greater than $w * k$. Therefore, the original shortest path doesn't change.

3 Shortest Paths Algorithm Design

Two countries, Mondstadt and Fontaine, are located in the fictional world of Teyvat. The railroad system of Teyvat can be modeled as a **weighted directed graph**, with V vertices, E edges, and weights being the length of the railway. Circle the traveler wishes to take railway from Mondstadt to Fontaine, and needs to determine the shortest railway distance between them. Define the set M to be all cities in Mondstadt, and F to be all cities in Fontaine. The shortest distance between the two countries is the shortest distance between any city c_M in Mondstadt and c_F in Fontaine.

For each of the subparts below, describe an algorithm that compute the minimum railway distance from Mondstadt to Fontaine, in $O((V+E)\log V)$ time. You are able to call all graph algorithms you learned in class as a black box. *Hint: For some parts, consider modifying the graph so that running a graph algorithm yields an equivalent answer to solving the original problem.*

(a) Mondstadt only contains 1 city ($|M| = 1$), but Fontaine contains many cities ($|F| > 1$).

Solution: Run Dijkstra's, starting from the only city m in M. This will generate the shortest distance to all cities of Teyvat. Then, iterate over every city f in F, and take the minimum distance from m to f.

(b) Mondstadt contains many cities ($|M| > 1$), but Fontaine only contains one city ($|F| = 1$).

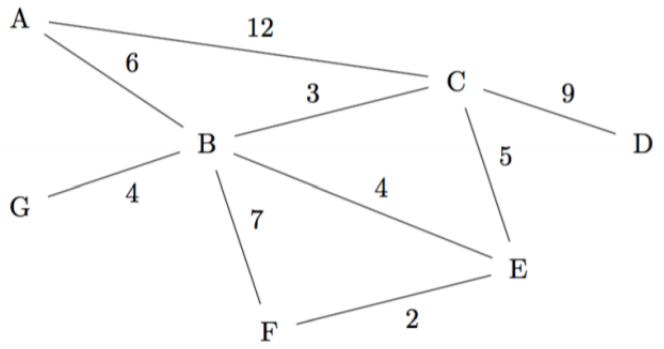
Solution: Create a dummy vertex d. For each city in M, create an edge from d to that city, with edge weight 0. This creates a new graph G'. Run Dijkstra's, starting from dummy vertex d, on the graph G'. Take the distance from d to the only city in F and return it. This works because for each path in the original graph from any city in M to F, it corresponds to a path in the new graph G' that starts from d and ends in F, and they are equal in length. Therefore, our algorithm correctly computes the shortest path from any city in M to any city in F.

(c) Both countries contain many cities ($|M|, |F| > 1$).

Solution: These steps are the same as previous part: Create a dummy vertex d. For each city in Mondstadt, create an edge from d to that city, with edge weight 0. This creates a new graph G'. Run Dijkstra, starting from dummy vertex d, on the graph G'.

Now, instead of directly taking the distance from d to the only city in F as in the previous part, take the minimum distance from d to all cities in F, and return it.

4 Introduction to MSTs



(a) For the graph above, list the edges in the order they're added to the MST by Kruskal's and Prim's algorithm. Assume Prim's algorithm starts at vertex A. Assume ties are broken in alphabetical order. Denote each edge as a pair of vertices (e.g. AB is the edge from A to B).

Prim's algorithm order:

Kruskal's algorithm order:

Solution: Prim's algorithm order: AB, BC, BE, EF, BG, CD
Kruskal's algorithm order: EF, BC, BE, BG, AB, CD

(b) True/False: Adding 1 to the smallest edge of a graph G with unique edge weights must change the total weight of its MST.

Solution: True, either this smallest edge (now with weight +1) is included, or this smallest edge is not included and some larger edge takes its place since there was no other edge of equal weight. Either way, the total weight increases.

(c) True/False: If all the weights in an MST are unique, there is only one possible MST.

Solution: True, the cut property states that the minimum weight edge in a cut must be in the MST. Since all weights are unique, the minimum weight edge is always unique, so there is only one possible MST.

(d) True/False: The shortest path from vertex u to vertex v in a graph G is the same as the shortest path from u to v using only edges in T, where T is the MST of G.

Solution: False, consider vertices C and E in the graph above. The shortest path between C and E uses the edge CE, but it is not part of the MST of the graph.

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